Available online at www.sciencedirect.com
SCIENCE DIRECT®
International Journal of Thermal Sciences 44 (2005) 685–693
International Journal of Thermal Sciences
www.elsevier.com/locate/ijts

Analysis of thermal irreversibilities in a homogeneous and isotropic solid

Carlo Bartoli *

Department of Energetica “Lorenzo Poggi”, Faculty of Engineering, University of Pisa, Via Diotisalvi 2, 56126 Pisa, Italy

Received 2 September 2004; received in revised form 18 October 2004; accepted 8 February 2005

Abstract

This research study focuses its attention on the local and temporal distribution of the entropy production generated by thermal irreversibilities as well as on the fact that they contribute, together with the entropy flow through the frontier, towards raising the solid entropy, under situations of thermal transient. For simplicity, a simple unidimensional model, the semi-infinite solid, for which several solutions depending on the Fourier equation time are well known, is investigated; they are here discussed with respect to their aspects of second principle. Moreover, it should be noticed that both entropy and entropy production are not linear in temperature. The principle of superposition is, therefore, not valid for them; this gets the remarks expounded in this paper to be considered as an exemplification of the entropic problems in simple cases, rather than a method for working out entropic solutions in more complex cases.

Keywords: Entropy; Entropy production; Solid; Square-wave thermal excitation; Sharp thermal contact

1. General remarks

The object of this paper is to study the irreversibilities, due to heat transfer, occurring in a homogeneous and isotropic solid utilizing the finite time thermodynamic methods [1–6]. Thermal irreversibilities have been already taken into consideration in the past, particularly as regards their influence on mechanical performance (thermal friction) of thermodynamic systems [7–11]. Let us remind you that, for a homogeneous, isotropic solid standing idle, with constant thermal conductivity and without internal heat production, the thermal conduction equation (Fourier equation) can be written as follows:

$$\frac{\partial T}{\partial t} - \alpha \nabla^2 T = 0 \tag{1.1}$$

As a matter of fact, the most common solutions require the thermal diffusivity α characterizing the solid properties in (1.1) to be a constant and its specific capacity per unit volume to be, then, constant too (the product of density and specific heat); hereinafter in this paper all the engine properties are assumed to be constants; in particular, the density ρ is assumed to be a constant. This assumption relating to the solid perfect non-deformability leads to a great simplification as regards the thermodynamic properties of the solid under investigation, assuring the absence of any expansion work, and then the equality between the specific heat at constant volume and that at constant pressure ($C_v = C_p = C$). More precisely, a solid portion seems to look like a incompressible fluid; the absence of work makes each heat acquisition increase the internal energy in the system, while both entropic flow on the contour and entropy production further the increase in entropy. Hereinafter by c ($c = \rho C$) specific capacity per unit volume will be indicated, by k thermal conductivity (therefore, $k = c\alpha$); by small letters the extensive thermodynamic properties referred to volume unit will be indicated; moreover, the internal energy per unit volume u and the specific entropy per unit volume s can be expressed (unless there are two arbitrary constants) as:

$$u = cT + \text{cost}, \quad s = c \ln T + \text{cost} \tag{1.2}$$

with T absolute temperature.

---

* Tel.: +39 (0)50 2217113; fax: +39 (0)50 2217150.
E-mail address: c.bartoli@ing.unipi.it (C. Bartoli).

1290-0729/$ – see front matter © 2005 Elsevier SAS. All rights reserved.
doi:10.1016/j.ijthermalsci.2005.02.002

Nomenclature

a	finite warmer solid thickness	m
d	finite colder solid thickness	m
c	specific thermal capacity per unit volume	$\text{J}\cdot\text{m}^{-3}\cdot\text{K}^{-1}$
b	thermal effusivity defined as, $=\sqrt{(ck)}$	$\text{J}\cdot\text{m}^{-2}\cdot\text{K}^{-1}\cdot\text{s}^{-0.5}$
H	dimensionless parameter defined as, $=\Delta T/T$	
k	thermal conductivity	$\text{W}\cdot\text{m}^{-1}\cdot\text{K}^{-1}$
L	total thickness defined as, $=a+b$	m
Q	heat	J
q	thermal specific flux	$\text{W}\cdot\text{m}^{-2}$
S	entropy	$\text{J}\cdot\text{K}^{-1}$
s	specific entropy per unit volume	$\text{J}\cdot\text{m}^{-3}\cdot\text{K}^{-1}$
T	temperature	K
t	time	s
t_r	relaxation time	s
t^*	normalized time	
U	internal energy	J
u	internal energy per unit volume	$\text{J}\cdot\text{m}^{-3}$
x	spatial coordinate	m
Y	dimensionless variable defined in Eq. (2.8)	
Z	dimensionless variable defined in Eq. (2.14″)	
Greek symbols		
α	thermal diffusivity	$\text{m}^2\cdot\text{s}^{-1}$
Λ	entropic production per unit of surface	$\text{J}\cdot\text{m}^{-2}\cdot\text{K}^{-1}$
λ	entropic production per unit of time and volume	$\text{J}\cdot\text{m}^{-2}\cdot\text{K}^{-1}$
Φ	quantity defined in Eq. (1.9)	$\text{J}\cdot\text{m}^{-2}$
ϕ	quantity defined in Eq. (1.4)	$\text{W}\cdot\text{m}^{-2}\cdot\text{K}^{-1}$
σ	characteristic time of problem	s
ψ	dimensionless variable defined in Eq. (2.13)	
Subscripts		
$+$	quantity refers to warmer solid	
$-$	quantity refers to colder solid	
Superscript		
$*$	dimensionless parameter	

The idealized solid will be supposed to occupy the half space $x>0$, and only the unidimensional solutions $T=T(x,t)$ of Eq. (1.1) will be taken into account; Eq. (1.1) can be, therefore, rewritten as follows:

$$\frac{\partial T}{\partial t}-\alpha\frac{\partial^2 T}{\partial x^2}=0 \tag{1.3}$$

If the heat fluxes $q(x,t)$ are introduced as well as the entropic ones $\phi(x,t)$

$$q=-k\frac{\partial T}{\partial x}, \qquad \phi=\frac{q}{T}=-\frac{k}{T}\frac{\partial T}{\partial x} \tag{1.4}$$

it is easy to discern that Eq. (1.3) identifies itself with the local formulation of the first principle

$$\frac{\partial u}{\partial t}+\frac{\partial q}{\partial x}=0 \tag{1.5}$$

Besides, resorting to some manipulations the following is also obtained:

$$\frac{\partial s}{\partial t}+\frac{\partial \phi}{\partial x}=\lambda$$

$$\text{where } \lambda=-\frac{q}{T^2}\frac{\partial T}{\partial x}=+k\left(\frac{1}{T}\frac{\partial T}{\partial x}\right)=\frac{\phi^2}{k} \tag{1.6}$$

representing the local formulation of the second principle; inside it the term $\lambda(x,t)$, always positive, represents the entropy production, per time and volume unit, quantifying the extent of the irreversibilities imputable to heat transfer.

The considered solutions of Eq. (1.3) refer to the case of perturbation of a preexistent balance state of the solid, by an excitation localized on the plane $x=0$ and acting from the time $t=0$; the initial temperature distribution inside the solid is a constant $T(x,t=0-)=T_i$, as well as uniform are the internal energy and the specific entropies, u_i and s_i. If by ΔU and ΔS the internal energy and entropy variations, per front surface unit of the solid, undergone by the whole solid between the starting instant $t=0-$ and the generic one t, are indicated

$$\Delta U(t)=\int_0^\infty\left[u(t)-u_i\right]\mathrm{d}x=c\int_0^\infty\left[T(t)-T_i\right]\mathrm{d}x$$

$$\Delta S(t)=\int_0^\infty\left[s(t)-s_i\right]\mathrm{d}x=c\int_0^\infty \ln\frac{T(t)}{T_i}\,\mathrm{d}x \tag{1.7}$$

the first and the second law of thermodynamics, in their integral formulation, can be respectively expressed by

$$\begin{aligned}&\Delta U(t)=Q_0(t)\\&\Delta S(t)=\Phi_0(t)+\Lambda(t)\end{aligned} \tag{1.8}$$

In Eq. (1.8), Q_0 and Φ_0 represent, respectively, the entropy and heat quantity coming into the solid from a unitary portion of the front surface, placed in $x=0$, from the initial instant to the generic one t:

$$Q_0(t)=\int_0^t q(0,t')\,\mathrm{d}t', \qquad \Phi_0(t)=\int_0^t \phi(0,t')\,\mathrm{d}t' \tag{1.9}$$

(the corresponding fluxes localized on the surface to infinity are, obviously, null). The term

$$\Lambda(t)=\int_0^t \mathrm{d}t'\int_0^\infty \lambda(x,t')\,\mathrm{d}x \tag{1.10}$$

represents the whole entropy production, originated in the whole solid between the initial instant and the generic one t and referred to the front surface unit of the solid itself. In other words, while the increase in internal energy is only caused by the heat Q_0 incoming from the front surface, the increase in entropy is made possible both by the entropic flux Φ_0 and the entropy production Λ.

2. Temperature with stepped increase

At first, the case in which the solid frontier temperature $x = 0$ is abruptly varied from the balance initial value T_i to a new constant value $T_i + \Delta T$ is considered [12,13]; it is well known that the solution of Eq. (1.3) according to the conditions

$$T(x,0) = T_i \quad \text{for } x \geqslant 0$$

$$T(0,t) = T_i + \Delta T \quad \text{for } t > 0 \tag{2.1}$$

can be expressed as follows

$$T = T_i + \Delta T \,\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \tag{2.2}$$

where the time scale σ is arbitrary and $\mathrm{erfc}(z)$ is the complementary the error function:

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x \tag{2.3}$$

From Eq. (2.2) entropic flux and heat flux are drawn:

$$q = \frac{k\Delta T}{\sqrt{\pi\alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right) \tag{2.4}$$

$$\phi = \frac{k\Delta T \exp(-\frac{x^2}{4\alpha t})}{\sqrt{\pi\alpha t}[T_i + \Delta T \,\mathrm{erfc}(\frac{x}{2\sqrt{\alpha t}})]} \tag{2.5}$$

while the entropy production just follows immediately from the last of Eq. (1.6):

$$\lambda = \frac{\phi^2}{k} \tag{2.6}$$

Entropy production space–time distribution. The previous relations allow to discuss the space and time distribution of the entropy sources. In Fig. 1 the dimensionless coordinate $x^* = \frac{x}{\sqrt{\alpha\sigma}}$ is reported in abscissa and the dimensionless entropic flux $\sqrt{\phi^*} = \sqrt{\frac{\sigma}{kc}\frac{T_i\phi}{|\Delta T|}}$ in ordinate (with σ problem-characteristic period that in this case may be chosen arbitrarily) for $H = \Delta T/T_i = 0.5$. This figure provides information about entropy production λ in that, according to Eq. (2.6), it is proportional to the square of ϕ. Analogously, Fig. 2 refers to the entropy flux per time and volume unit for $H = -0.5$. Let us distinguish the case in which $H > 0$, corresponding to a heating of the solid with q always positive, from that in which $H < 0$, where the solid cools and q is always negative; in both cases the remarkable values of q are localized near the front surface within a thickness of about some times $\sqrt{(\alpha t)}$: this implies that the entropy flux ϕ and the entropy production λ too have remarkable values within the same solid portion. Moreover, while at any given instant t, the module of q always decreases autonomously while penetrating into the solid, in the case $H < 0$ the quantities $|\phi|$ and λ behave analogously, but in the case $H > 0$ the higher H is the more evident the maximum shown by them are. In other words, even though they are localized within a thickness layer widening out in time, equalling about $\sqrt{(\alpha t)}$, the maximum entropy production keeps localized on the front surface in case of cooling, while, in case of heating, it penetrates into the solid. This penetration concerning the case $H > 0$ can be studied quantitatively, by seeking the value of X maximizing ϕ, for an assigned value of t: going to zero the derivative $\partial\phi/\partial x$, the following transcendental equation is obtained:

$$1 + H\,\mathrm{erfc}(Y) - \frac{H\exp(-Y^2)}{Y\sqrt{\pi}} = 0 \tag{2.7}$$

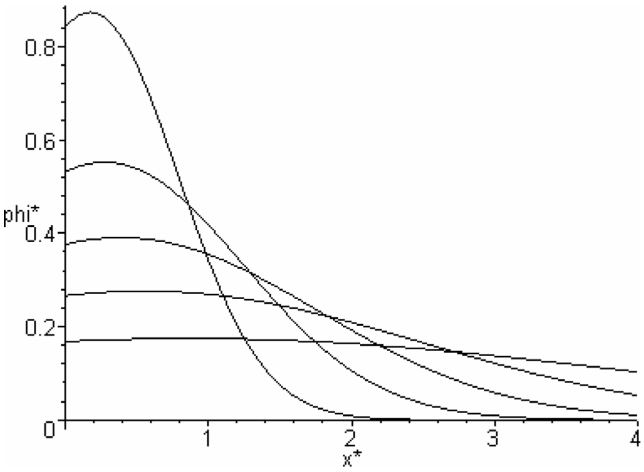

Fig. 1. Temperature with stepped increase: spatial trend of dimensionless entropic flux for $H = 0.5$ at $t^* = t/\sigma = 0.2$, 0.5, 1, 2, 5.

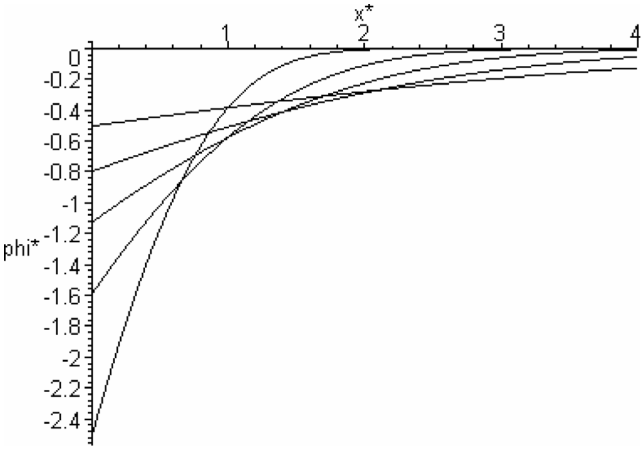

Fig. 2. Temperature with stepped increase: spatial trend of dimensionless entropic flux for $H = -0.5$ at times $t^* = t/\sigma = 0.2$, 0.5, 1, 2, 5.

in the dimensionless variable

$$Y = \frac{X}{2\sqrt{\alpha t}} \tag{2.8}$$

A numerical solution of Eq. (2.7) allows to deduce the function $Y(H)$: the abscissa for which the maximum entropy production occurs is the following:

$$X = 2Y(H)\sqrt{\alpha t} \tag{2.9}$$

As far as the solid global behaviour is concerned, the increase in internal energy till the instant t is completely determined by the heat quantity coming in from the front plane; according to the first of Eq. (1.9), it clearly turns out to be

$$\Delta U(t) = Q_0(t) = 2k\Delta T\sqrt{\frac{t}{\pi\alpha}} \tag{2.10}$$

Moreover, it rises in time proportionally to $\sqrt{t}$. Correspondingly, according to the second of Eq. (1.9), the incoming entropy

$$\Phi_0(t) = 2k\sqrt{\frac{t}{\pi\alpha}}\frac{\Delta T}{T_i + \Delta T} = \frac{1}{1+H}\frac{Q_0(t)}{T_i} \tag{2.11}$$

turns out to be proportional to the heat quantity. On the other hand, the increase in entropy, according to the second of Eq. (1.7), turns out to be the following:

$$\begin{aligned}\Delta S(t) &= c\int_0^\infty \ln\left[1 + H\,\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right]\mathrm{d}x \\ &= 2c\sqrt{\frac{\alpha t}{\pi}}\Psi(H)\end{aligned} \tag{2.12}$$

in which the quantity

$$\Psi(H) = \sqrt{\pi}\int_0^\infty \ln\bigl[1 + H\,\mathrm{erfc}(\zeta)\bigr]\,\mathrm{d}\zeta \tag{2.13}$$

Owing to the presence of the entropy production, in the case of heating ($H > 0$), the incoming entropy is not sufficient for determining the whole increase in entropy, while, in the case of cooling ($H < 0$), an entropy higher than the solid decrease in entropy comes out of the front surface. In any case, the relation between the quantities being here investigated

$$\frac{\Phi_0}{\Delta S} = Z(H) \quad \text{where } Z(H) = \frac{H}{(1+H)\Psi(H)} \tag{2.14}$$

depends on the parameter H only, and is independent both of time and solid physical characteristics. Besides, the total entropy production turns out to be (at each instant t) the following:

$$\Lambda = \Delta S - \Phi_0 = \Phi_0\left(\frac{1}{Z} - 1\right) \tag{2.15}$$

For instance, for a heating varying from 300 to 360 K ($H = 0.2$) we have $Z = 0.880$. The 88% of the solid increase in entropy is therefore imputable to the incoming entropy and the remaining 12% to the entropy production localized within the solid; moreover, connected with each $\Delta U = Q_0 = 1\ \mathrm{J{\cdot}m^{-2}}$ of energy absorbed by the solid, we have an incoming entropy $\Phi_0 = 2.78 \times 10^{-3}\ \mathrm{J{\cdot}m^{-2}{\cdot}K^{-1}}$, an increase in entropy $\Delta S = \Phi_0/Z = 3.15 \times 10^{-3}\ \mathrm{J{\cdot}m^{-2}{\cdot}K^{-1}}$ and an entropy production $\Lambda = \Delta S - \Phi_0 = 0.38 \times 10^{-3}\ \mathrm{J{\cdot}m^{-2}{\cdot}K^{-1}}$.

Obviously, for $t \to \infty$ it results $\Delta U \to \infty$ and $\Delta S \to \infty$.

3. Square-wave thermal excitation

In this section the excitation of a solid, caused by an increase ΔT of its surface temperature, kept for a time interval t_0 (which is the time peculiar to the problem: $\sigma = t_0$), is considered [14–16]. In other words, the following conditions are assumed:

$$\begin{aligned}&T(x,0) = T_i \quad \text{for } x \geqslant 0\\ &T(0,t) = T_i + \Delta T \quad \text{for } 0 < t < t_0\\ &T(0,t) = T_i \quad \text{for } t_0 \leqslant t\end{aligned} \tag{3.1}$$

This excitation turns out to be the superposition of an excitation of the type shown in Section 2 switched on at the instant $t = 0$ and of a second excitation, of opposite sign, switched on at the instant $t = t_0$; for the Fourier equation linearity the sought solution turns out to be the following:

$$T = \begin{cases} T_i + \Delta T\,\mathrm{erfc}\bigl(\frac{x}{2\sqrt{\alpha t}}\bigr) \\ \quad \text{for } 0 < t < t_0 \\ T_i + \Delta T\bigl[\mathrm{erfc}\bigl(\frac{x}{2\sqrt{\alpha t}}\bigr) - \mathrm{erfc}\bigl(\frac{x}{2\sqrt{\alpha(t-t_0)}}\bigr)\bigr] \\ \quad \text{for } t \geqslant t_0 \end{cases} \tag{3.2}$$

In particular, for $t < t_0$ the temperature trend coincides with Eq. (2.2): it follows that also the expressions of the various quantities ($q, \phi, \lambda, \ldots$) are identical to those reported in Section 2: hereinafter the expressions for $t > t_0$ only are, therefore, reported. In particular, according to thermal power, we have for:

$$\begin{aligned} q = \frac{k\Delta T}{\sqrt{\pi\alpha}}\biggl[&\frac{1}{\sqrt{t}}\exp\left(-\frac{x^2}{4\alpha t}\right) \\ &- \frac{1}{\sqrt{t-t_0}}\exp\left(-\frac{x^2}{4\alpha(t-t_0)}\right)\biggr] \quad \text{for } t \geqslant t_0 \end{aligned} \tag{3.3}$$

From these relations follow immediately those relating to ϕ and λ.

Entropy production space–time distribution. The previous relations allow to discuss the space and time distribution of the entropy sources. In Figs. 3 and 4 the dimensionless entropic flux is reported, respectively for $H = 0.5$ and $H = -0.5$. For a qualitative examination, $\Delta T > 0$ is considered at first. While for each $0 < t < t_0$ the temperature profile in the solid decreases monotonically with x, passing from the surface value $T_i + \Delta T$ to the asymptotic value T_i, for $t > t_0$ it increases from T_i till it reaches a maximum peak

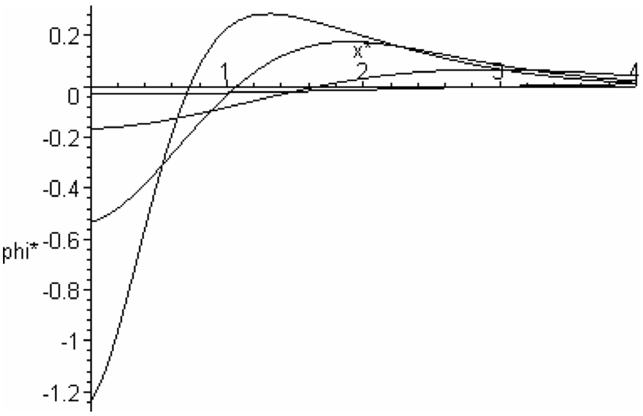

Fig. 3. Square-wave thermal excitation: spatial trend of dimensionless entropic flux for $H = 0.5$ at times $t^* = t/\sigma = 0.2, 0.5, 1, 2, 5$.

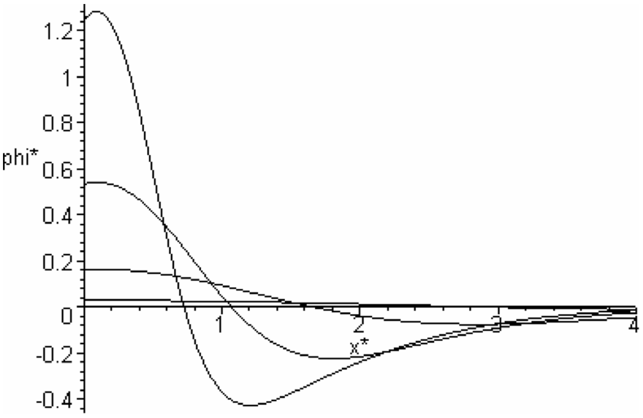

Fig. 4. Square-wave thermal excitation: spatial trend of dimensionless entropic flux for $H = -0.5$ at times $t^* = t/\sigma = 0.2, 0.5, 1, 2, 5$ (cooling).

in a position x_0, and then decreases again. Correspondingly, the flux q is everywhere positive (and decreasing) in the first case, while, in the second case, it results to be negative for $x < x_0$ and positive for $x > x_0$, with a maximum in x_m: the values of the positions x_0 and x_m vary in time according to the following relations:

$$x_0^2 = \frac{2\alpha t(t-t_0)}{t_0}\big[\ln(t) - \ln(t-t_0)\big]$$
$$x_m^2 = \frac{6\alpha t(t-t_0)}{t_0}\big[\ln(t) - \ln(t-t_0)\big] \tag{3.4}$$

Moreover, the entropic flux ϕ, as discussed in Section 2, shows a maximum peak for $t < t_0$, while for $t > t_0$ it shows a qualitative trend similar to that of q, with a zero in x_0 and a maximum near a value a little higher than x_m. The entropy production λ, while for $t < t_0$ shows a maximum peak in X, for $t > t_0$ decreases, at first, monotonically till it annuls itself in x_0 and then reaches a maximum peak (having a value inferior to the surface value) near x_m. For $\Delta T < 0$ the entropic flux for $t > t_0$ shows a maximum peak near the origin and a minimum peak for an x a little lower than x_m; two distinct entropy production maximum peaks correspond to them. In any case, the entropy production seems to be actually localized, for the most part, within the surface layer having thickness x_0.

As far as the solid global behaviour is concerned, the increase in internal energy till the instant t is still given by Eq. (2.10) if $t < t_0$, while the following is valid:

$$\begin{aligned}\Delta U(t) &= Q_0(t)\\ &= \frac{2k\Delta T}{\sqrt{\pi\alpha}}\big[\sqrt{t} - \sqrt{t-t_0}\big] \quad \text{for } t \geqslant t_0\end{aligned} \tag{3.5}$$

In other words, if $\Delta T > 0$, the increase in energy assumes the maximum value at the end of the impulse, where it assumes the following value:

$$\Delta U(t_0) = Q_0(t_0) = 2k\Delta T\sqrt{\frac{t_0}{\pi\alpha}} \tag{3.6}$$

and then decreases tending to zero as time increases. The incoming entropy is still given by Eq. (2.11) if $t < t_0$, while it results to be the following for $t \geqslant t_0$

$$\Phi_0(t) = \frac{2k\Delta T}{\sqrt{\pi\alpha}}\left[\frac{\sqrt{t_0}}{T_i + \Delta T} - \frac{\sqrt{t-t_0} - \sqrt{t} + \sqrt{t_0}}{T_i}\right] \tag{3.7}$$

Therefore, if $\Delta T > 0$, it rises during the excitation till it reaches the maximum value

$$\Phi_0(t_0) = \frac{2k\Delta T}{T_i + \Delta T}\sqrt{\frac{t_0}{\pi\alpha}} \tag{3.8}$$

in $t = t_0$, and then decreases monotonically, going to zero at the instant

$$t_z = t_0\left[\frac{1 + 2H + 2H^2}{2H(1+H)}\right]^2 \tag{3.9}$$

Vice versa, for $\Delta T < 0$, the entropic flux keeps always negative. Finally, for $t \to \infty$ it tends to the asymptotic value

$$\begin{aligned}\Phi_\infty &= -\frac{2k\Delta T^2}{T_i(T_i + \Delta T)}\sqrt{\frac{t_0}{\pi\alpha}}\\ &= -Q_0(t_0)\left[\frac{1}{T_i + \Delta T} - \frac{1}{T_i}\right]\end{aligned} \tag{3.10}$$

The increase in entropy is expressed by Eq. (2.12) if $t < t_0$, while the following is valid

$$\begin{aligned}\Delta S(t) = 2c\sqrt{\alpha t}\int_0^\infty \ln\Bigg\{1 + H\Bigg[&\operatorname{erfc}(\zeta)\\ &- \operatorname{erfc}\left(\zeta\sqrt{\frac{t}{t-t_0}}\right)\Bigg]\Bigg\}\,\mathrm{d}\zeta \quad \text{for } t \geqslant t_0\end{aligned} \tag{3.11}$$

This quantity keeps always higher than $\Phi_0(t)$ and assumes its maximum value at the end of the excitation, and then decreases tending to zero for $t \to \infty$; in particular, Eq. (3.10) shows an entropic flux coming out from the solid, due to a total entropy production $\Lambda_\infty = -\Phi_\infty$, corresponding to the heat quantity $Q_0(t_0)$ passing from a temperature $T_i + \Delta T$ to T_i.

In Figs. 5 and 6 the temporal trends of the entropy variation, of the entropic flux in the origin and of the total entropy production defined as

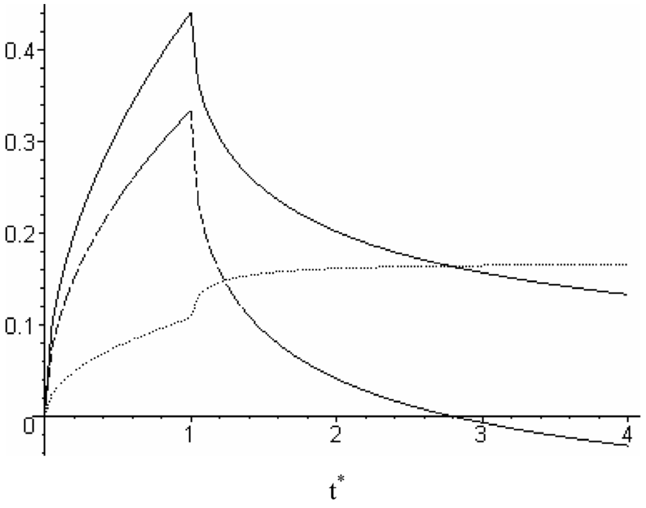

Fig. 5. Square-wave thermal excitation: Trends of spatial distribution of entropy (solid line), of entropic flux in the origin of the axes (dashed line) and of entropy production (dotted line) vs time for $H = 0.5$; all the quantities are normalized.

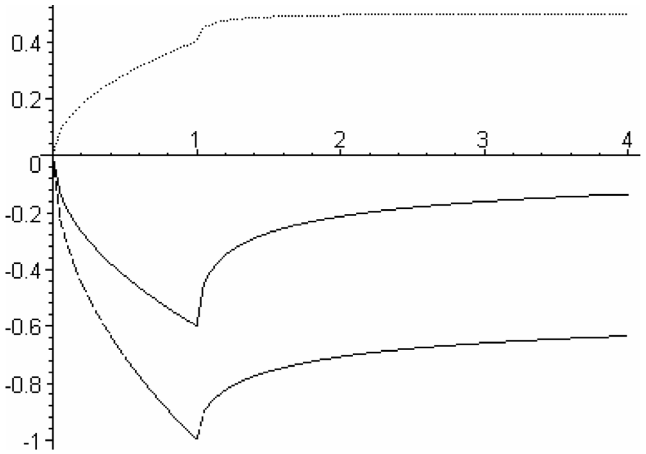

Fig. 6. Square-wave thermal excitation: Trends of spatial distribution of entropy (solid line), of entropic flux in the beginning of axes (dashed line) and of entropy production (dotted line) vs time, for $H = -0.5$; all the quantities are normalized.

$$\Delta S^* = \frac{\Delta S}{2c}\sqrt{\frac{\pi}{\alpha\sigma}}, \qquad \Phi_0^* = \frac{\Phi_0}{2c}\sqrt{\frac{\pi}{\alpha\sigma}}$$

$$\Lambda^* = \frac{\Lambda}{2c}\sqrt{\frac{\pi}{\alpha\sigma}}$$

are represented in a normalized form both in case of increase in temperature ($\Delta T > 0$) and in case of decrease in temperature ($\Delta T < 0$); while for $t < t_0$ these quantities vary with the square root of time, keeping proportional to each other (as in the case of Section 2), for $t > t_0$ they show a more complex behaviour. In particular, it can be noticed that a short while after the impulse end ($t^* \approx 2$) almost all the entropy production has fully developed, being close to its asymptotic value, while the entropy keeps on varying (with little dissipation) due to the entropic flux through the origin.

4. Sharp thermal contact between two semi-infinite solids

The above-obtained results can be used in the following problem [17–20]: at the instant $t = 0$ let us bring into perfect thermal contact two solids (schematized as two half spaces), the first keeping, at the beginning, its balance at the temperature T_+, the other at the temperature T_-. Let the two solids have distinct thermal properties, marked by the pedices $+$ and $-$; moreover, $T_+ > T_-$ can be considered, the hotter body can be localized in the half space $x > 0$ and the colder one in the $x < 0$, without losing generality. The thermal transient trend can be obtained by solving the Fourier equation for the two solids, imposing the above-stated initial conditions as well as the equality conditions of temperatures and heat fluxes on the separation plane $x = 0$.

It is easy to check the temperature trend

$$T = \begin{cases} T_+ + (T_\infty - T_+)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha_+ t}}\right) & \text{for } x > 0 \\ T_- + (T_\infty - T_-)\,\mathrm{erfc}\left(\frac{-x}{2\sqrt{\alpha_- t}}\right) & \text{for } x < 0 \end{cases} \tag{4.1}$$

to satisfy all the above-listed requirements, on condition that the following expression is chosen for the asymptotic temperature

$$T_\infty = \frac{T_- b_- + T_+ b_+}{b_- + b_+} \tag{4.2}$$

with b the effusivity, defined as $b = \sqrt{ck}$, representing an average weighing of the initial temperatures. In other words, both solids undergo a thermal development of the type studied in Section 2, with a temperature jump in the axes origin to the asymptotic value T_∞ (reached abruptly in $t = 0+$). In the formulae of Section 2 the following should be meant:

$$\begin{aligned} H_- &= \frac{T_\infty - T_-}{T_-} = \frac{(T_+ - T_-)b_+}{T_-(b_- + b_+)} \\ H_+ &= \frac{T_\infty - T_+}{T_+} = \frac{(T_+ - T_-)b_-}{T_+(b_- + b_+)} \end{aligned} \tag{4.3}$$

which, according to the above-stated conditions, turn out to be, the first positive, the second negative. In particular, the heat quantity passing through the origin within the time t can be written as follows:

$$\begin{aligned} Q_0(t) &= -2b_+\sqrt{\frac{t}{\pi}}(T_+ - T_\infty) = 2b_-\sqrt{\frac{t}{\pi}}(T_\infty - T_-) \\ &= -2\sqrt{\frac{t}{\pi}}\frac{b_- b_+}{b_- + b_+}(T_+ - T_-) \end{aligned} \tag{4.4}$$

having negative sign since it has retrograde direction compared to the axis x; clearly, the heat quantity absorbed by the solid in $x < 0$ has the opposite sign and this is the reason why in the formulae concerning fluxes the sign is to be changed for this solid. Besides, the entropy transferred in $x = 0$ by a solid is completely absorbed by the other, that is:

$$\Phi_0(t) = \frac{Q_0(t)}{T_\infty} = -2\sqrt{\frac{t}{\pi}}\frac{(T_+ - T_-)b_- b_+}{T_- b_- + T_+ b_+} \tag{4.5}$$

As far as thermal irreversibilities are concerned, the maximum entropy production of the hotter solid is always localized in $x = 0$, while for the colder one it penetrates into the solid, according to the (2.9), i.e.

$$X = -2Y(H_-)\sqrt{\alpha_- t} \tag{4.6}$$

while, as far as the total entropy production in the two solids is concerned, we have, respectively:

$$\Lambda_- = -\Phi_0\left(\frac{1}{Z(H_-)} - 1\right)$$
$$\Lambda_+ = \Phi_0\left(\frac{1}{Z(H_+)} - 1\right) \tag{4.7}$$

As an example, the case in which $T_+ = 360$ K, $T_- = 300$ K is going to be considered with the two solids made of the same material: it follows that $T_\infty = 330$ K, $H_- = 1/10$, $H_+ = -1/12$, $Z(H_-) = 0.9352$, $Z(H_+) = 1.0639$. For $Q_0 = -1$ J·m^{-2} (i.e., for each Joule transferred from the area unit of the hot solid to the cold one) an analogous entropy transfer occurs $\Phi_0 = Q_0/T_\infty = -3.03 \times 10^{-3}$ J·m^{-2}·K^{-1}, with entropy productions $\Lambda_- = -\Phi_0 0.0692 = 0.21 \times 10^{-3}$ J·m^{-2}·K^{-1}, $\Lambda_+ = \Phi_0(-0.0600) = 0.18 \times 10^{-3}$ J·m^{-2}·K^{-1}. A slightly higher entropy production can be noticed in the initially colder $\Lambda_-/(\Lambda_- + \Lambda_+) = 0.54$, equalling the 54% of the total amount.

Holding the same initial temperatures ($T_+ = 360$ K, $T_- = 300$ K), let us consider two solids with different properties and, more exactly, with $b_+/b_- = 2$; $T_\infty = 340$ K, $H_- = 2/15$, $H_+ = -1/18$, $Z(H_-) = 0.9160$, $Z(H_+) = 1.0414$ are calculated. For $Q_0 = -1$ J·m^{-2} an entropy transfer $\Phi_0 = Q_0/T_\infty = -2.94 \times 10^{-3}$ J·m^{-2}·K^{-1} occurs, with entropy productions $\Lambda_- = -\Phi_0 0.0917 = 0.27 \times 10^{-3}$ J·m^{-2}·K^{-1}, $\Lambda_+ = \Phi_0(-0.0398) = 0.12 \times 10^{-3}$ J·m^{-2}·K^{-1}. Notice that the highest entropy production occurs in the solid situated on the left, where a dissipation equalling $\Lambda_-/(\Lambda_- + \Lambda_+) = 0.70$ occurs.

Then the positions of the two solids are exchanged, but keeping that situated on the right ($x > 0$) initially always hotter: i.e., the case in which $T_+ = 360$ K, $T_- = 300$ K with $b_+/b_- = 1/2$ is considered; notice that, time being equal, the exchanged heat quantity turns out to be the same as in the previous case, but the exchanged entropy turns out to be different. $T_\infty = 320$ K, $H_- = 1/15$, $H_+ = -1/9$, $Z(H_-) = 0.9556$, $Z(H_+) = 1.0876$ are calculated. For $Q_0 = -1$ J·m^{-2} an analogous entropy transfer occurs $\Phi_0 = Q_0/T_\infty = -3.13 \times 10^{-3}$ J·m^{-2}·K^{-1}, with entropy productions $\Lambda_- = -\Phi_0 0.0465 = 0.15 \times 10^{-3}$ J·m^{-2}·K^{-1}, $\Lambda_+ = \Phi_0(-0.0805) = 0.25 \times 10^{-3}$ J·m^{-2}·K^{-1}; in this case the highest entropy production occurs in the solid situated on the right, where the 63% of the total dissipation takes place. Therefore, the highest entropy dissipation clearly seems to occur in the solid with the lowest value of effusivity, even if it is that being, initially, hotter.

5. Finite solid

In the previous sections it has been pointed out how, due to a thermal perturbation imposed on a solid surface, the consequent thermal transient leads to an entropy production exhausting itself soon in the course of time, concerning zones of the solid being about $\sqrt{(\alpha t)}$ in depth; more precisely, it seems to be a quite general rule that in connection with an abrupt heating of the solid front surface, the maximum entropy production is just localized on the solid perturbed surface, while in connection with an abrupt cooling, the maximum of the instantaneous entropy production penetrates, fading drastically, into the solid itself.

From a global point of view, most of the entropy production develops during the first instants, as, for instance, is quantified in Section 3, where Λ can be observed to approximate its asymptotic value after few units of normalized time. Moreover, the treatment of the previous sections, even though it simplifies the entropy production due to abrupt thermal transients, says nothing on the achievement of thermal balance; in Section 4 too, this balance is not achieved because of the infinite dimensions assumed for the two solids brought into contact. If, more realistically, two finite-dimensional solids are considered, a "relaxation time" t_r is introduced for the balance achievement; only for very short times, compared to t_r, the above-considered entropic properties are obtained due to the abrupt thermal transient. In order to analyze also the entropy production occurring during the relaxation towards the balance (for times of about t_r), the bringing into thermal contact of two portions of the same material, provided with plane symmetry and isolated at the free extremities, is considered; in this case we have a simple solution of the Fourier equation allowing an easy discussion of the problem.

In this section the case of two solid layers being respectively a and $d = L - a$ in thickness, with a and d being equal in their size, is examined: therefore, the relaxation time t_r turns out to be about L^2/β. From a qualitative point of view it can be deduced that, for very short times compared to t_r, the thermal development is that considered in Section 4: the temperature of the contact point abruptly jumps to the value $T_\infty = (T_+ + T_-)/2$, arithmetical mean of the temperature values of the solid's two portions, and keeps practically constant, while the temperature perturbation tends to concern both the solid's parts. Farther on, when the perturbation reaches the extremities, phenomena of subsequent reflections of the perturbation itself on the adiabatic extremities occur with a consequent temperature change in the whole solid (and then in the contact point as well); the balance, final temperature T_{eq}, which is reached asymptotically at the end of the process, can be easily assessed according to a simple energy balance

$$T_{\mathrm{eq}} = \frac{a}{L}T_- + \frac{d}{L}T_+ \tag{5.1}$$

At the same time, the whole solid undergoes a global entropy variation equalling

$$\begin{aligned}\Delta S_{\mathrm{eq}} &= cL\ln(T_{\mathrm{eq}}) - ca\ln(T_-) - cd\ln(T_+)\\ &= CL\ln\left[\frac{\frac{a}{L}T_- + \frac{d}{L}T_+}{T_-^{a/L} + T_+^{d/L}}\right] \qquad (5.2)\end{aligned}$$

entirely attributable (owing to the process adiabaticity) to the entropy production by thermal transfer.

More exactly, a detailed solution of the thermal trend can be achieved considering the well-known solution of the Fourier equation [21] for an indefinite plane wall with thickness L, adiabatic at the two extremities, with an arbitrary initial temperature profile $f(x)$

$$\begin{aligned}T &= \frac{1}{L}\int_0^L f(x')\,\mathrm{d}x' + \frac{2}{L}\sum_{n=1}^{\infty}\exp\left(-\frac{n^2\pi^2\alpha t}{L^2}\right)\\ &\quad\times\cos\left(\frac{n\pi x}{L}\right)\int_0^L f(x')\cos\left(\frac{n\pi x'}{L}\right)\mathrm{d}x' \qquad (5.3)\end{aligned}$$

Notice that in this section, unlike the previous, the origin of the axis x is assumed to coincide with the solid left extremity. Putting in Eq. (5.3) the temperature initial profile

$$f(x) = \begin{cases} T_- & \text{for } 0 < x < a\\ T_+ & \text{for } a < x < L\end{cases} \qquad (5.4)$$

characterizing this problem, the required solution is obtained

$$\begin{aligned}T(x,t) &= T_- + (T_+ - T_-)\left[\left(1 - \frac{a}{L}\right)\right.\\ &\quad - \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\left(\frac{n\pi a}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\\ &\quad\left.\times\exp\left(-\frac{n^2\pi^2\alpha t}{L^2}\right)\right] \qquad (5.5)\end{aligned}$$

the heat flux is

$$\begin{aligned}q(x,t) &= -\frac{2k(T_+ - T_-)}{L}\\ &\quad\times\sum_{n=1}^{\infty}\sin\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2\alpha t}{L^2}\right) \qquad (5.6)\end{aligned}$$

Fig. 7 points out the contact temperature temporal development in agreement with the above-reported qualitative discussion; moreover, it shows the process part, characterized by contact phenomena, to last just some hundredth of the length peculiar to the problem, while the equilibrium condition is reached, in practice, after some times $t^* = \alpha T/L$. This figure shows that in the first instants after the thermal contact the temperature coincides with the arithmetic mean with value 1.25. Then this value develops and tends to the equilibrium temperature (values 1.375 and 1.125, respectively); the solution of infinite solid appears reasonable until t^* is of the order of a few hundredth.

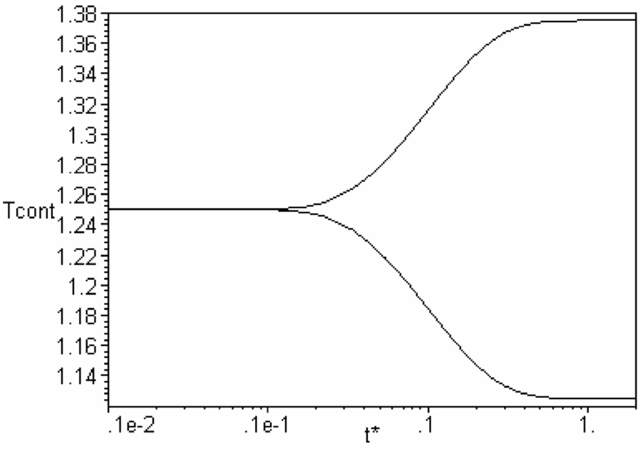

Fig. 7. Trend of normalized contact temperature [$T\mathrm{cont} = T(x=0)/T_-$] between two parts of solid in function of normalized time for $a/L = 0.25$ (upper curve) and for $a/L = 0.75$ (down curve)

It can be interesting to study the entropy temporal development; in particular, it can be noticed that it tends “very rapidly” to the limit value (5.2); for this purpose the following entropy variation is reported:

$$\begin{aligned}\Delta S(t) &= c\int_0^L \ln(T)\,\mathrm{d}x - ca\ln(T_-) - cd\ln(T_+)\\ &= cL\int_0^1 \ln\left(\frac{T}{T_{\mathrm{eq}}}\right)\mathrm{d}\left(\frac{x}{L}\right) - ca\ln\left(\frac{T_-}{T_{\mathrm{eq}}}\right)\\ &\quad - cd\ln\left(\frac{T_+}{T_{\mathrm{eq}}}\right) = cLJ + \Delta S_{\mathrm{eq}}\end{aligned}$$

with

$$J = \int_0^1 \ln\left(\frac{T}{T_{\mathrm{eq}}}\right)\mathrm{d}\left(\frac{x}{L}\right)$$

always negative quantity, describing the entropy increase deviation from its asymptotic value ΔS_{eq}; an analysis of the integral, not to be reported hereinafter, shows the term fading more slowly than J to be

$$-\frac{1}{\pi^2}\left(\frac{T_+ - T_-}{T_{\mathrm{eq}}}\right)^2\sin^2\left(\frac{\pi a}{L}\right)\exp\left(-\frac{2\pi^2\alpha t}{L^2}\right)$$

(fading as the square of the first term in the above-reported expansion of q).

Besides, during the initial phenomenon of bringing into contact, the increase in entropy can be approximated, according to the considerations of Sections 2 and 4, summing the increases in entropy of the solid’s two parts, according to Eq. (2.12); we, therefore, have:

$$\frac{\Delta S(t)}{cL} \approx \frac{2\sqrt{t}}{\sqrt{\pi t_r}}\left[\Psi\left(\frac{T_\infty - T_-}{T_-}\right) + \Psi\left(\frac{T_\infty - T_+}{T_+}\right)\right]$$

Fig. 8 represents the trend of dimensionless entropy increase $\Delta S/(cL)$ vs $t^* = t/t_r$ in the range between 0.001 and 1;

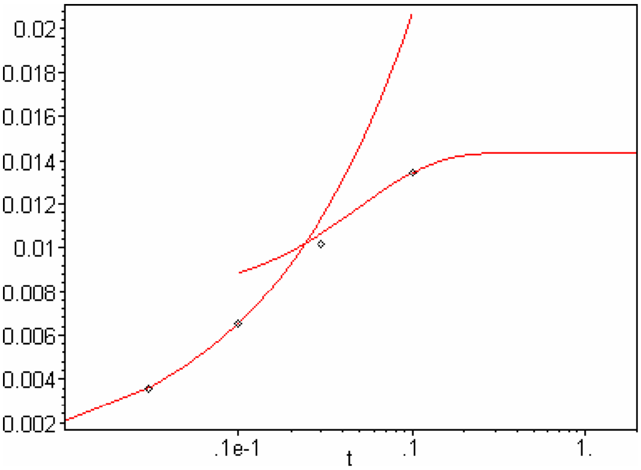

Fig. 8. Trend of entropy increment vs time for t^* from 0.001 to 1.

the two curves refer to the approximate formulas and more precisely the formula (5.9) is relative to smaller values of t^* whereas Eq. (5.8) to higher values of t^*. Lastly the points have been calculated with the exact formula (5.7).

6. Conclusions

The irreversibilities due to heat transfer in a homogeneous and isotropic solid are investigated using the Fourier equation as well as the first and the second laws of thermodynamics, starting, initially, from semi-infinite bodies with stepped increase in temperature using the function erf. Moreover, the square-wave thermal excitation has been taken into consideration studying in particular the space–time distribution of the entropy production. Subsequently, the possibility of bringing into thermal contact two solids, always schematized as two half spaces, at different temperature has been considered, and the most of the entropy production has been discovered to occur within the body with the lower product between specific thermal capacity per unit volume and thermal conductivity. Finally, two finite-dimensional solids have been considered with comparable thickness and thermally isolated from the outside: in this case, after bringing into contact the two bodies at different temperature, the entropy variation has been analyzed during the transient in which the thermal perturbation concerning both bodies produces, once it has reached the adiabatic extremities, subsequent reflections, leading the bodies to reach the same final temperature.

References

[1] H.B. Callen, Thermodynamics and an Introduction to Thermostatistics, Wiley, New York, 1985.

[2] M.C. Mackey, The dynamic origin of increasing entropy, Rev. Mod. Phys. 61 (1989) 981–990.

[3] S. Sieniutycz, P. Salomon (Eds.), Finite-Time Thermodynamics, Taylor & Francis, London, 1990.

[4] E.P. Gyftopoulos, G.P. Beretta, Thermodynamics: Foundation and Applications, Macmillan, New York, 1991.

[5] C. Bartoli, M. Ciampi, G. Tuoni, Analisi di irreversibilità termiche interne nell'ambito della termodinamica in tempo finito, in: Proceedings of the 13th UIT National Conference, Bologna, Italy, 1995, pp. 79–90.

[6] Z. Kolenda, J. Donizak, J. Hubert, On the minimum entropy production in steady state heat conduction processes, Energy 29 (2002) 2441–2460.

[7] A. Kodal, B. Shain, T. Yilmaz, A comparative performance analysis of irreversible Carnot heat engine under maximum power conditions, Energy Convers. Management 41 (2000) 235–248.

[8] E.P. Gyftopoulos, On the Curzon F.L. and Alborn B. efficiency and its lack of connection to power producing processes, Energy Convers. Management 43 (2002) 609–615.

[9] F.L. Curzon, B. Alborn, Efficiency of Carnot engine at maximum power output, Amer. J. Phys. 43 (1975) 22–24.

[10] C. Bartoli, M. Ciampi, G. Tuoni, Ciclo di Carnot generalizzato, in: Proceedings of the 11th UIT National Conference, Milano, Italy, 1993, pp. 441–452.

[11] M. Ciampi, F. Leccese, G. Tuoni, Carnot heat engine at maximum power output. Heat transfer with a fluid subject to periodic temperature variation, in: Proceedings of the 4th European Thermal Sciences Conference, Birmingham, United Kingdom, 2004, pp. 441–452.

[12] M.N. Ozisik, Heat Conduction, Wiley, New York, 1980.

[13] S. Faggiani, Complementi di Fisica Tecnica, ETS Editrice, Pisa, 1989.

[14] V.S. Arpaci, Conduction Heat Transfer Conduction Heat Transfer, Addison-Wesley, Reading, MA, 1966.

[15] H. Gzyl, The Method of Maximum Entropy, World Scientific, Singapore, 1995.

[16] A. Bejan, Entropy Generation Minimisation, CRC Press, Boca Raton, FL, 1966.

[17] A.V. Luikov, Analytical Heat Diffusion Theory, Academic Press, New York, 1968.

[18] J. Bransier, Sur le role de l'effusivité dans le conctacts thermiques brefs, Internat. J. Heat Mass Transfer 17 (7) (1974) 727–738.

[19] J.P. Bardon, Heat transfer at solid–solid interface. Basic phenomena. Recent work, in: Proceedings of the 4th Eurotherm, Nancy, France, vol. 1, 1988, pp. 40–74.

[20] T. Loulou, J.P. Bardon, Premiers instants du refroidissement d'une goutte métallique après son impact sur une paroi, Rev. Gen. Therm. 36 (1997) 682–689.

[21] H.S. Carslaw, J.C. Jaeger, Conduction of Heat in Solid, Oxford University Press, New York, 1975.